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Population Growth and Industrial Revolution Cycles

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Abstract

History shows that industrial revolutions come in waves; since the latter half of the 18th century we have observed an industrial revolution at least three times over the past three 270 years. This study incorporates population growth in a Matsuyama model of innovation cycles and investigates whether industrial revolution cycles may be caused by population growth. Our results show that industrial revolution cycles may occur if the population growth rate is set at about 1 to 1.2 percent and if the other parameter values are set consistently with real world data. The average population growth rate was about 0.9 percent in the UK during the period covering the first industrial revolution and the first half of the second industrial revolution. It was about 1.2 percent in the US during the period covering the latter hald of the second industrial revolution through the third industrial revolution. The long-term population growth rates of the other regions are lower than those UK and US rates.

Keywords: industrial revolution cycles, population growth, innovation, market quality

1 Introduction

History shows that industrial revolutions come in waves; since the latter half of the 18th century we have observed what is called an industrial revolution at least three times. Even during a period of industrial revolution, new and influential technologies come in waves. A century ago, Nikolai Kondratieff (1926, 1935) discovered similar but shorter innovation cycles of 50 to 60 year period, which he attributed to some deterministic economic mechanism; his idea was followed by Scumpeter (1939). Building a simple macroeconomic model, Yano and Furukawa (2023) reveal that those long waves of Kondratieff can be explained as a part of longer innovation cycles, which they refer to as industrial revolution cycles. In that study, this phenomenon is explained by the interaction between R&D activities (innovation) and an exogenous increase in labor productivity.

In this study, we investigate whether industrial revolution cycles may be caused by population growth rather than labor productivity increase. In the early literature on growth, productivity increase and population growth are interchangeable (Solow, 1956). In more sophisticated modern macroeconomic models, however, they are not. An infinitely-lived agent model, like that of Yano and Furukawa (2023), is not suitable for dealing with population growth. In the Matsuyama model (Matsuyama, 1999), in which endogenous growth is incorporated into the Solow model, population growth is incompatible with endogenous growth.

This study develops a simple macroeconomic model of the Matsuyama type, in which a simple Keynesian saving function is assumed; this assumption is justifiable, given our focus on the First and Second Industrial Revolutions.

In order to incorporate population growth into the Matsuyama model, we introduce strict concavity into the production function of the final consumption good; in contrast, Matsuyama (1999) assumes that the production is linearly homogeneous, which excludes the possibility of harmoneous growth in endogenous and exogenous growth factors.

The main finding of this study is that to observe industrial revolution cycles of about 100 year period in our model, the population growth rate has to be relatively high within a realistic range in our model, provided that the other parameters of the model are set consistently with real-world data during the First and Second Industrial Revolution. To demonstrate this result, we adopt Yano and Furukawa's characterization of two-dimensional ergodic chaos; erogodicity refers to the property that the time average of a solution to a deterministic dynamical system (time average) can be characterized by the space average or a probability distribution; see Birkoff (1932) and von

Neumann (1932). Lasota and Yorke (1973) demonstrate that a single dimensional dyanmical system is ergodic if it is unimodal and expansive. Yano and Furukawa (2023) demonstrate that in a two-dimensional dynamical system the domain of which is subject to an exogenous constraint (constrained chaos), double period solutions obey an ergodically chaotic system of the Lasota-Yorke type.[1] We characterize the ergodically chaotic region of parameter values, which is impossible in Yano and Furukawa's model, and demonstrate that for a range of parameter values consistent with real world data, industrial revolution cycles of about 100 year period are observed in the model.

The existence of constrained chaos is observed in a single dimensional dynamical system by Nishimura and Yano (1994, 1995a, 1995b, 1999). Their results are extended by Baierla, Nishimura, Yano (1998), Mitra and Khan (2005), Khan and Piazza (2011), and Deng and Khan (2018).

Kondratieff-Schumpeter's idea of innovation cycles has been picked up by a number of theoretical studies, which explain innovation cycles as a deterministic phenomenon. See Judd (1985), which Yano and Furukawa (2023) extend; moreover, see Shleifer (1986), Gale (1996), and Matsuyama (1999, 2001), and Francois and Lloyd-Ellis (2003). The existence of chaotic dynamics in the models of Judd (1985) and Matsuyama (1999) is studied extensively; see Deneckere and Judd (1992) for the Judd model and Mitra (2001), Mukherji (2005), and Yano, Sato, and Furukawa (2011) for the Matsuyama model. For a study of the role of labor in chaotic equilibrium, see Iong and Irmen (2021). See Boldrin, Nishimura, Shigoka and Yano (1998) for chaotic dyanmics in an endogenous growth model.

Our study on industrial revolution cycles, as well as Yano and Furukawa (2023), complements the literature on innovation-driven growth (Romer, 1990, and Rivera-Batiz and Romer, 1991). More broadly, it is also related to the literature on endogenous growth (Grossman and Helpman (1991), and Aghion and Howitt (1992)).

In what follow, we present our model in Section 2. In Section 3, we transform the model into a two-dimensional dynamical system and derive a set of conditions under which the system is ergodically chaotic. In Section 4, we characterize industrial revolution cycles. Proofs for all the theorems and lemmas (except Theorem 3) are given in a separate appendix.

[1] That follows from Lasota and Yorke (1973), Kowalski (1975), and Li and Yorke (1978); see Bhattacharya and Majumdar (2007) and Grandmont (2008) for details.

2 Basic Model

Population grows at a constant rate, $\lambda > 1$. Following the standard literature, a constant portion of population is fully employed every period. Thus, the amount of labor employed in period $t+1$ can be written as

$$L_{t+1} = \lambda L_t, \quad t = 0, 1, ..., \tag{1}$$

where $L_0 = \bar{L}$ is given.

As in Matsuyama (1999, 2001), we distinguish capital goods and capital, the latter of which is transformed into the former. Capital goods are differentiated and produced by using differentiated technologies. New technologies can be invented every period, given economically viable. Let N_t be the number of technologies available in petiond t. The technology for the i-th capital good is invented before period t if $i < N_{t-1}$ and invented in period t if $N_{t-1} \leq i < N_{t-1}$. Let $x_t(i)$ be the amount of capital good produced by the i-th technology. If no new technologies are invented in period t, $N_{t-1} = N_t$ and $x_i(i) = 0$. As in the standard literature (Judd, 1985, Matsuyama, 1999, and Yano and Furukawa, 2023), we assume that the number of technologies monotonically increases over time, i.e.,

$$N_t - N_{t-1} \geq 0. \tag{2}$$

The final goods sector produces the final goods (or income), Y_t, from capital goods $x_t(i)$, $0 \leq i \leq N_t$, and labor L_t,

$$Y_t = \left(\int_0^{N_t} [x_t(i)]^\alpha di \right)^\beta L_t^\gamma, \tag{3}$$

where

$$0 < \alpha < 1, \ 0 < \beta < 1, \ \text{and } \beta + \gamma = 1. \tag{4}$$

Since $\beta + \gamma = 1$, the production technology for the final goods is of constant returns to scale with respected to technologies (or the number of differentiated technologies) and labor. However, it can be of decreasing returns to scale (if $\alpha < 1$) with respect to the volume of capital goods input.

To develop a new technology, $i \in [N_{t-1}, N_t)$, its developer has to use a fixed amount of capital F for research and development. Following (Solow, 1956), moreover, we assume that the capital goods (differentiated), $x_t(i)$, are from the final consumption good produced in the previous period. We assume a constant fraction of is a constant fraction of the income, μY_{t-1}. These relationships imply

$$\int_0^{N_{t-1}} a x_t(i) di + \int_{N_{t-1}}^{N_t} (a x_t(i) + \varphi) di = \mu Y_{t-1}, \tag{5}$$

where a is the amount of capital needed to produced one unit of a capital good.

As in Matsuyama (1999), the final goods sector is assumed to be perfectly competitive. In aggregate, it maximizes its surplus,

$$\Pi_t = (\int_0^{N_t} [x_t(i)]^\alpha di)^\beta L_t^\gamma - \int_0^{N_t} p_t(i)x_t(i)di - w_t L_t, \tag{6}$$

where $p_t(i)$ is the price of the i-th capital good and where w_t is the wage rate; note that the price of the final consumption good is normalized to be equal to 1. The first order condition for profit maximization gives rise to the following inverse demand function, relating each individual capital good, $x_t(i)$, to the market price, $p_t(i)$, i.e.,

$$p_t(i) = \alpha\beta \left(\int_0^{N_t} (x_t(i)^\alpha di) \right)^{-(1-\beta)} {L_t}^\gamma x_t(i)^{-(1-\alpha)}. \tag{7}$$

The developer of a new technology, i, uses a fixed amount of capital φ to invent a new technology. Following Judd (1985), Deneckere and Judd (1991), Matsuyama (1999), and Yano and Furukawa (2023), we assume that the developer is allowed to use its technology monopolistically to sell its output just for one period. From the next period on, the technology becomes freely used by anyone.

Each technology transforms α units of capital into one unit of a capital good. The profit of a technology developer is

$$\pi_t(i) = p_t(i)x_t(i) - \{ax_t(i) + \varphi\}r_t, \quad i \in [N_{t-1}, N_t]. \tag{8}$$

The developer, as a monopolist of its product, faces the (inverse) demand function for its product, (7). Following the standard literature, we assume that developer $i \in [N_{t-1}, N_t)$ takes the macro variables ($\int_0^{N_{t-1}} [x_t(i)]^\alpha di$ and L_t to be precise) as given and chooses its supply (which is equal to demand $x_t(i)$) so as to maximize the profit, $\pi_t(i)$, by assuming that price $p_t(i)$ is determined by supply $x_t(i)$ through (7). This gives rise to the following first order condition for profit maximization,

$$\alpha p_t(i) = ar_t, \quad i \in [N_{t-1}, N_t). \tag{9}$$

Following, again, the standard literature, we assume free entry for the innovation sector (of technology developers), which leads each developer's profit to zero, i.e., $\pi_t(i) = 0$ for $i \in [N_{t-1}, N_t)$, which results in

$${x_t}^m = \frac{\alpha}{a(1-\alpha)}\varphi. \tag{10}$$

If $i \in [0, N_{t-1})$, as is noted above, capital good i is supplied perfectly competitively in period t. Thus, its price, $p_t(i)$, is equal to the marginal cost, ar_t, i.e.,

$$p_t(i) = ar_t, \quad i \in [N_{t-1}, N_t). \tag{11}$$

$$p_t(i) = \frac{a}{\alpha} r_t,$$

This implies that $p_t(i) = p_t$ is independent of $i \in [0, N_{t-1})$, which implies $x_t(i) = {x_t}^c$. This completes the description of our model.

Normalize the amounts of capital and the number of technologies into par capita variables,

$$k_t = K_t/L_t \text{ and } n_{t-1} = N_{t-1}/L_t. \tag{12}$$

Then, we may demonstrate that an equilibrium in the above model follow the following an autonomous dyanmical system.

Theorem 1 *Our model is summarized by the following two-phase dynamical system with* $k_t = K_t/L_t \geq 0$ *and* $n_{t-1} = N_{t-1}/L_t \geq 0$:

$$n_t \geq \frac{1}{\lambda} n_{t-1}; \tag{13}$$

If $n_{t-1} < \alpha^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{F} k_t$,

$$\begin{pmatrix} k_{t+1} \\ n_t \end{pmatrix} = \begin{pmatrix} \frac{\mu}{\lambda} \left(\frac{\eta}{1-\alpha}\right)^{\beta(1-\alpha)} \left(\frac{\alpha}{a}\right)^{\alpha\beta} {k_t}^{\beta} \\ \frac{1}{\lambda} \frac{1-\alpha}{\eta} k_t + \frac{1-\alpha^{-\frac{\alpha}{1-\alpha}}}{\lambda} n_{t-1} \end{pmatrix}; \tag{14}$$

If $n_{t-1} \geq \alpha^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{F} k_t$,

$$\begin{pmatrix} k_{t+1} \\ n_t \end{pmatrix} = \begin{pmatrix} \frac{\mu}{\lambda} \left(\frac{\varphi}{1-\alpha}\right)^{\beta(1-\alpha)} \left(\frac{\alpha}{a}\right)^{\alpha\beta} {k_t}^{\beta} \\ \frac{1}{\lambda} \frac{1-\alpha}{\varphi} k_t + \frac{1-\alpha^{-\frac{\alpha}{1-\alpha}}}{\lambda} n_{t-1} \end{pmatrix}. \tag{15}$$

Matsuyama (1999) shows that, in his model, a state variable alternates to lie in two phases, which he calls Romer and Solow regimes. In our model, the Romer regime is characterized by $n_{t-1} < \alpha^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{\varphi} k_t$, , in which innovation takes place. The Solow regime is characterized by $n_{t-1} \geq \alpha^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{\varphi} k_t$ since, for example, $n_t < \lambda y_t$ implies $n_t < \lambda n_{t+1}$, which means that the number of new technologies developed in period t is $N_t - N_{t-1} > 0$.

3 Two-dimensional Chaotic System

As is shown below, the unique feature of our model is that the two-dimensional economic dynamical system of per capita capital and per capita technology can be dycotomized into a two-dimensional system with per capita technology and an associated system determining the dynamics of per capita capital with the per capita technology variable acting as a parameter (see Theorem 2). To apply their result, it is useful to transform the above model into a simple two-dimensional dynamical system. Towards this end, define

$$y_t = \begin{cases} \frac{1}{\lambda}\frac{1-\alpha}{F}k_{t+1} + \frac{1-\alpha^{-\frac{\alpha}{1-\alpha}}}{\lambda}n_t & \text{if} \quad n_t < \alpha^{\frac{\alpha}{1-\alpha}}\frac{1-\alpha}{F}k_{t+1} \\ \frac{1}{\lambda}n_t & \quad\quad n_t \geq \alpha^{\frac{\alpha}{1-\alpha}}\frac{1-\alpha}{F}k_{t+1}; \end{cases} \tag{16}$$

as this expression shows, variable y_t aggregates per capita capital k_{t+1} and per capita technology n_t.

Theorem 2 *Let*

$$\kappa = \frac{\mu}{\lambda^{2-\beta}}\left(\frac{F}{1-\alpha}\right)^{\beta(2-\alpha)-1}\left(\frac{\alpha}{a}\right)^{\alpha\beta} \tag{17}$$

and

$$\varphi = \alpha^{-\frac{\alpha}{1-\alpha}}. \tag{18}$$

The vector of per present and next period per capita technologies, (n_t, y_t) *obeys the constrained two dimensional system where*

$$\begin{aligned} \begin{pmatrix} n_t \\ y_t \end{pmatrix} &= \begin{pmatrix} y_{t-1} \\ \max\{\kappa\left(y_{t-1} - \frac{1-\varphi}{\lambda}n_{t-1}\right)^{\beta} + \frac{1-\varphi}{\lambda}y_{t-1}, \frac{1}{\lambda}y_{t-1}\} \end{pmatrix} \\ &\equiv f\begin{pmatrix} n_{t-1} \\ y_{t-1} \end{pmatrix}. \end{aligned} \tag{19}$$

3.1 Double-period System

The system in Theorem 2 has a structure similar to Yano and Furukawa's system of double-period chaos (2023). They derive the basic conditions under which the constrained dynamics like that of Theorem 2 gives rise to a double-period dynamical system with a single variable

$$n_{2\tau} = \underbrace{T\circ\cdots\circ}_{\tau\text{ times}}T(n_0) = T^{(\tau)}(n_0). \tag{20}$$

If this double-period system, T, is expansive and unimodal, it is ergodically chaotic. Unfortunately, however, Yano and Furukawa's model is too complicated to determine a set of parameter values under which T is double-period chaos. The second main result of this study is to build a model simple enough to obtain such a set; in contrast, in Yano and Furukawa (2023), only the existence of parameter values with which T is ergodically chaotic is demonstrated.

Define

$$\ell_0(n_{-1}) = \lambda n_{-1} \tag{21}$$

and, construct inductively a sequence of functions

$$n_{\tau+1} = \ell_{\tau+1}(n_\tau), \tag{22}$$

$\tau = 0, 1, 2, ..,$ by

$$n_\tau = \ell_\tau(n_{\tau-1}) \text{ and } n_{\tau+1} = \kappa \left(n_\tau + \frac{\varphi - 1}{\lambda} n_{\tau-1} \right)^\beta - \frac{\varphi - 1}{\lambda} n_\tau. \tag{23}$$

To state our result, think of the following double period dynamical system. With

$$L(n) = \ell_3 \circ \ell_2(n) \text{ and } R(n) = \ell_1 \circ \ell_0(n), \tag{24}$$

define

$$T(n_{2(t+1)}) = \min\{L(n_{2t}), R(n_{2t})\}. \tag{25}$$

3.2 Constrained Chaos in Two dimension

To explain the global structure of our constrained two-dimensional system, F, Figure 1 is useful. Recall $\ell_0(n) = \frac{1}{\lambda} n$, the graph of which is line ℓ_0, starting from the origin with slope $1/\lambda < 1$. By (19), any solution to constrained two-dimensional system $(n_t, y_t) = f(n_{t-1}, y_{t-1})$ must lie above or on line ℓ_0. We may say that the state variable obeys interior dynamics above this line and boundary dynamics on the line. As the next lemma shows, the space of state variables (n, y) is divided into two half spaces by curve $n = b(y)$,

$$b(y_t) = -\frac{\lambda}{\varphi - 1} y_t + \frac{\lambda}{\varphi - 1} \left(\frac{\varphi}{\lambda \kappa} \right)^{1/\beta} y_t^{1/\beta} \tag{26}$$

Lemma 1 *Function $b(y)$ is convex, increasing and, thus, invertible for $b(y) > 0$. Moreover,*

$$\begin{aligned} y_t &= \tfrac{1}{\lambda} n_t \quad \text{if } y_{t-1} \geq b^{-1}(n_{t-1}) \\ y_t &> \tfrac{1}{\lambda} n_t \quad \text{if } y_{t-1} < b^{-1}(n_{t-1}). \end{aligned} \tag{27}$$

In Figure 1, the graph of the inverse function, $y = b^{-1}(n)$, is shown by curve b^{-1}. The state variable on or above curve b^{-1} must reach line ℓ_0 in the next period; that below curve b^{-1} reach in the interior of the feasible set, or a point above line ℓ_0. If the state variable that lies above line ℓ_0 must eventually leave the interior dynamics region, and if it lies on line ℓ_1 must eventually leave the boundary dynamics region, it will indefinitely wander around between interior and boundary dynamics regions. Yano and Furukawa (2023) shows that this dynamics could be ergodic (Birkoff (1932) and von Neumann (1933)). This study captures a parameter region in which Yano and Furukawa's theorem holds; it is impossible to obtain such a region in Yano and Furukawa's model due to the complexity of their model.

3.3 Diagrammatic Exposition

As is explained in the previous subsection, if the state variable cannot be confined within the core of constrained dyanamics, or the region between ℓ_0 and b^{-1}, it will return to line ℓ_0, and therfore, every solution to system (19) can be chacterized by examining solutions starting from points on ℓ_0. Suppose that dynamics goes though a point on ℓ_0, (n_{t-1}, n_t) with $n_t = \frac{1}{\lambda} n_{t-1}$ to a point above ℓ_0. This is characterized by $\ell_1(n_t)$, which is, by (21)-(23),

$$n_{t+1} = \ell_1(n_t) = \kappa\varphi^\beta n_t^\beta - \frac{\varphi - 1}{\lambda} n_t. \tag{28}$$

Lemma 2 *Functions $\ell_1(n)$ is differentiable and concave. On $n > 0$, it takes its maximum,*

$$n_1 = \kappa\varphi^\beta \left(\frac{\lambda\beta\kappa\varphi^\beta}{\varphi - 1}\right)^{\frac{\beta}{1-\beta}} - \frac{\varphi - 1}{\lambda} \left(\frac{\lambda\beta\kappa\varphi^\beta}{\varphi - 1}\right)^{\frac{1}{1-\beta}}, \tag{29}$$

at $n = \left(\frac{\lambda\beta\kappa\varphi^\beta}{\varphi-1}\right)^{\frac{1}{1-\beta}}$ and is decreasing on interval $I_1 = [n_1, \infty)$, where

$$n_1 = \left(\frac{\lambda\beta\kappa\varphi^\beta}{\varphi - 1}\right)^{\frac{1}{1-\beta}}. \tag{30}$$

By (21)-(23), ℓ_2 is given as

$$n_{t+2} = \ell_2(n_{t+1}) = \kappa \left(n_{t+1} + \frac{\varphi - 1}{\lambda} \ell_1^{-1}(n_{t+1})\right)^\beta - \frac{\varphi - 1}{\lambda} n_{t+1}. \tag{31}$$

Lemma 3 *Functions $\ell_2(n)$ is differentiable, concave and decreasing on interval $I_2 = [n_2, \infty)$, where*

$$n_2 = \kappa \left(\kappa\varphi^\beta n_1\right)^\beta - \frac{\varphi - 1}{\lambda}\left[\kappa\varphi^\beta n_1^\beta - \frac{\varphi - 1}{\lambda} n_1\right]. \tag{32}$$

On I_2, it takes its maximum,

$$n_2 = \ell_2(0) = \kappa \left(\frac{\varphi - 1}{\lambda}\left(\frac{\lambda\kappa\varphi^\beta}{\varphi - 1}\right)^{\frac{1}{1-\beta}}\right)^\beta. \tag{33}$$

Moreover, on the interval,

$$\ell_2'(n) < \ell_1'(n). \tag{34}$$

By (21)-(23), again, ℓ_3 is given as

$$n_{t+3} = \ell_3(n_{t+2}) = \kappa \left(n_{t+2} + \frac{\varphi - 1}{\lambda}\ell_2^{-1}(n_{t+2})\right)^\beta - \frac{\varphi - 1}{\lambda} n_{t+2}. \tag{35}$$

Lemma 4 *Functions $\ell_3(n)$ is differentiable, concave and decreasing on interval $I_3 = (-\infty, n_3]$*

$$n_3 = \kappa \left(\kappa \left(\frac{\varphi - 1}{\lambda} n_1\right)^\beta\right)^\beta. \tag{36}$$

Moreover, on the interval,

$$\ell_3(n) < \ell_2(n). \tag{37}$$

See Figure 2, in which the graphs of $y = \ell_i(n)$, $i = 0, 1, 2, 3, 4$, are depited by curves ℓ_i, $i = 1, 2, 3, 4$, for the case in which T is ergodic chaos. Theorem 3 below implies that double period system T can be illustrated by the kinked graph, X, Y, and Z. We will first explain the three critical points of this graph, X, Y and Z.

Lemma 5 *Let n_B and n_C be given by the following:*

$$n_B = \lambda^{\frac{\beta}{1+\beta}} \left(\frac{\varphi - \lambda^{\frac{1}{1+\beta}}}{\varphi - 1}\right) \lambda^{\frac{1}{1-\beta^2}} \kappa^{\frac{1}{1-\beta}} \varphi^{-1}; \tag{38}$$

$$n_C = \lambda^{\frac{1}{1-\beta^2}} \kappa^{\frac{1}{1-\beta}} \varphi^{-1}. \tag{39}$$

Then, it holds that

$$n_C = \ell_1(n_B) \text{ and } \ell_0(n_C) = \ell_2(n_C). \tag{40}$$

Note that if (n_t, y_t) follows system (21), $(n_{\tau+1}, y_{\tau+1})$ lies on curve ℓ_1 if (n_τ, y_τ) lies on curve ℓ_0, on curve ℓ_2 if (n_τ, y_τ) lies on curve ℓ_1, and on curve ℓ_3 if (n_τ, y_τ) lies on curve ℓ_2. Let $n_D = \ell_0(n_C) = \ell_2(n_C)$ in period $t+1$. Then, in period $t+2$, the state variable must be at the intersection between curves ℓ_1 and ℓ_3, $(n_t, n_{t+1}) = (n_D, y_D) = D$. In other words,

$$y_D = \ell_1 \circ \ell_0(n_C) = \ell_3 \circ \ell_2(n_C), \tag{41}$$

which implies that $(n_{t-1}, n_{t+1}) = (n_C, y_D)$ is at Y.

By construction, the state variable from any point on curve b^{-1} reaches a point on line ℓ_0. By the definition of function b, moreover, curve b^{-1} goes through point $B = (n_B, y_B)$ from which point $C = (n_C, y_C)$ is reached in the next period, or $y_B = n_C$ and $y_C = \frac{1}{\lambda} n_C$. Since curve b^{-1} is increaing whereas curve ℓ_2 is decreasing, the intersection between curve b^{-1} and curve ℓ_2 must reach the intersection between line ℓ_0 and curve ℓ_3. By Lemma 4, this intersection lies below point C, as is shown in Figure 5.

Both curves b^{-1} and ℓ_2 go through point $B = (n_B, y_B)$. Since pont C is below B on ℓ_1, D must lie above curve b^{-1}. This implies that, by Lemma 1, the state variable $(n_{t+1}, n_{t+2}) = (n_E, y_E) = E$ must lie on line ℓ_0. In the subsequent period, therefore, the state variable must lie on curve ℓ_1, i.e., $n_{t+3} = \ell_1(n_{t+2}) = \ell_1 \circ \ell_0(n_{t+1})$; in other words, $n_{t+2} = y_F = n_F$,

$$y_F = \ell_1 \circ \ell_0(n_E). \tag{42}$$

Thus, $(n_{t+1}, n_{t+3}) = (n_E, y_F)$ lies at point Z.

Since the state variable lies on curve ℓ_1, i.e., $n_{t+3} = \ell_1(n_{t+2})$, and since $(n_{t+2}, n_{t+3}) = (n_F, y_F)$ lies below curve b^{-1}, in the next period, the state variable starting from (n_C, y_C) will lie on ℓ_2, say, at $G = (n_G, y_G)$; or $(n_{t+3}, n_{t+4}) = (n_{t+3}, \ell_2(n_{t+3})) = G$. Note that point $I = (n_I, n_I)$ is the intersection between curve ℓ_1 and the 45 degree line, i.e., $n_I = \ell_1(n_I)$. Since $y_F > n_I$, by Lemma 3, $\ell_2(n_G) < \ell_1(n_G) < n_I$. This implies $G = (n_G, y_G)$ lies on ℓ_2 below curve b^{-1}. Since $n_{t+4} = \ell_3(n_{t+3}) = \ell_3 \circ \ell_2(n_{t+2})$,

$$y_H = \ell_3 \circ \ell_2(n_G)). \tag{43}$$

Thus, $H = (n_H, y_H) = (n_{t+4}, n_{t+5})$ is on ℓ_3, $n_{t+5} = \frac{1}{\lambda} y_{t+4}$. Point $(n_{t+3}, n_{t+5}) = (n_G, y_H)$ is at X.

The discussion above shows that in system F, the state variable, starting from point C, moves to D and, then, E, G, H. In the double-period system, T, the state variable starting from point $Y = (n_C, y_D)$ and shifts to point Z and, then, point X. In order to prove the theorem, it is necessary to demonstrate that the state variable starting from any point n_{t-1}, $n_C \leq n_{t-1} \leq n_E$, follows the double-peroid system, T.

3.4 Characterization of Chaos

The next theorem characterizes the region of parameter values with which system f is ergodically chaotic.

Theorem 3 *Let n_B and n_C be given by (38) and (39). Moreover, define $n_A = \lambda n_B$, $n_D = \ell_0(n_C) = \ell_2(n_C)$, $n_E = \ell_1(n_D) = \ell_3(n_D)$, $n_F = \frac{1}{\lambda} n_E$, $n_G = \ell_2(n_F)$ and $n_H = \ell_3(n_G)$. Double period system $T : [n_D, n_E] \to [n_D, n_E]$ is an ergodic chaos that is continuous and satisfies $L'(n) > 1$ and $R'(n) < 1$ if the following holds:*

$$\ell_2(n_2) > \ell_0(n_2); \tag{44}$$

$$y_E < y_I; \tag{45}$$

$$n_B \geq n_H; \tag{46}$$

$$b^{-1}(n_C) \geq \frac{1}{\lambda} n_C; \tag{47}$$

$$b^{-}(n_E) \geq \frac{1}{\lambda} n_E; \tag{48}$$

$$\ell_2(n_G) < b^{-1}(n_G) \tag{49}$$

$$\ell_3(n_H) > b^{-1}(n_H) \tag{50}$$

$$\ell_1'(n_D) < -\lambda; \tag{51}$$

$$\ell_2'(n_G) < -\frac{1}{\lambda}. \tag{52}$$

Proof: Function $\ell_2(n)$ is decreasing on I_2 whereas $\ell_0(n)$ is increasing on $n \geq 0$. Thus, by (44), they intersect each other on I_2. The intersection is point $C = (n_C, n_D)$.

Take any n_{t-1}, $n_C \leq n_{t-1} \leq n_E$ and $n_t = \ell_0(n_{t-1}) = \frac{1}{\lambda} n_{t-1}$. Since b^{-1} is concave by Lemma 1, $n_C \leq n_{t-1} \leq n_E$ implies $n_t < b^{-1}(n_{t-1})$ by (47)/ and (48). Thus, by Lemma 1, $n_{t+1} = \ell_1(n_t) = \ell_1(\ell_0(n_{t-1})) = R(n_{t-1})$. Since, by (51), $\ell_1'(n_D) < -\lambda$, by Lemma 2, $\ell_1'(n_t) < -\lambda$. Thus,

$$R'(n_{t-1}) = \ell_1'(n_t)\ell_0'(n_{t-1}) = \frac{1}{\lambda}\ell_1'(n_t) < \frac{1}{\lambda}(-\lambda) = -1. \tag{53}$$

Since $n_C \leq n_{t-1} \leq n_E$, either $n_C \leq n_{t-1} \leq n_A$ or $n_A < n_{t-1} \leq n_E$.

Take the case of $n_C \leq n_{t-1} \leq n_A$. Then, $n_D \leq n_t = \ell_0(n_{t-1}) \leq n_B$ and $n_C \leq n_{t+1} = \ell_1(n_t) \leq n_E$. This implies $n_{t+1} = \ell_1(n_t) \geq b^{-1}(n_t)$. Thus, $n_C \leq n_{t+1} \leq n_E$. Thus, it follows from the same discussion as above that $n_{t+2} = \ell_0(n_{t+1})$.

Take the case of $n_A \leq n_{t-1} \leq n_E$. Then, $n_B < n_t = \ell_0(n_{t-1}) \leq n_F$, which implies $n_G \leq n_{t+1} = \ell_1(n_t) < n_C$. Since, by (49), $\ell_2(n_G) < b^{-1}(n_G)$, $n_G \leq n_{t+2} = \ell_2(n_{t+1}) < n_C$ it holds that $n_{t+2} = \ell_2(n_{t+1})$. Moreover, by (50), $\ell_3(n_H) > b^{-1}(n_H)$, $n_{t+2} > b^{-1}(n_{t+1})$, which implies $n_{t+3} = \ell_2(n_{t+2}) = \ell_3(\ell_2(n_{t+1})) = L(n_{t+1})$. Since ℓ_2 and ℓ_3 are concave, and since $n_{t+1} \geq n_G$ and $n_{t+2} > n_D$, $\ell_3'(n_{t+2}) \leq \ell_3'(n_D)$ and $\ell_2'(n_{t+1}) \leq \ell_2'(n_G)$. Moreover, by Lemma 4 and (51), $\ell_3'(n_D) < \ell_1'(n_D) < -\lambda$ and, by (52), $\ell_2'(n_G) < -1/\lambda$. Thus,

$$L'(n_{t+1}) = \ell_3'(n_{t+2})\ell_2'(n_{t+1}) \geq \ell_3'(n_D)\ell_2'(n_G) > 1. \tag{54}$$

This shows that any solution to system f from $(y_{-1}, n_{-1}) = (\lambda n, n)$ with $n_C \leq n \leq n_E$ satisfies $n_G \leq n_t \leq n_E$ and obeys $n_{t+1} = T(n_{t-1})$ for all t.

4 Industrial Revolution Cycles

Here, we investigate the frequency of industrial revolutions by setting the values of parameters in a range consistent with real world data. Our characterization of an industrial revolution follows Yano and Furukawa (2023) in focusing on what they call a technological burst, which is said to occur when the state variable falls in an interval $n_{t+1} \in [n_A, n_E]$; as is explained in that study, the pattern of dynamics is a distinct feature if it starts from that interval.

The number of new technologies developed in a period, t, is given by

$$\Delta N_t = N_{t+1} - N_t = \lambda^{t+1}(1 - 1/\lambda)n_t, \tag{55}$$

which captures the innovation level in t. The innovation level jumps from zero to the maximum possible level if $n_{t-1} = n_C$. In this case, $n_{t+1} \in [n_A, n_E]$. With this consideration, Yano and Furukawa (2023) say that a big burst in innovation occurs in period t if an equilibrium path that falls in interval $[n_A, n_E]$ at the end of period t, $n_{t+1} \in [n_A, n_E]$ and regard a big burst as an industrial revolution. Such an equilibrium path distinct features of an industrial revolution period (see Yano and Furukawa, 2023).

The main focus of this study is on the effect of population growth on industrial revolution cycles. Table 1 shows the frequency of industrial revolution cycles for different parameter values that are consistent with real world data; recall that our model is governed by six parameters, λ, β, α, μ, F, and a. Of those F and a do not affect innovation cycles and are set at arbitrary levels, $F = 0.00001$ and $a = 0.0001$. Although data are not ample for the days of the first and second industrial revolutions, we set values of λ,

β, α, and μ consistent with real-world data as much as possible to analyze the frequency of industrial revolution cycles for each of the past and current industrial revolutions.

We set the model's population growth rate, λ, at various levels that are of interest in consideration of real-world facts. During the fist industrial revolution through the middle of the second, the UK is the leader of innovation. According to Broadberry, Campbell, Klein, Overton, and Leeuwen (2011, Table 10) and Maddison (2023), during that period, the UK population growth rate is about 0.90 percent.[2] In contrast, the leader shifted to the US around the very end of the 19th century, and the US population growth rate is 1.25 percent during the period from 1900 through 2022. With these considerations, we adopt $\lambda_o = 1.010$ and 1.012, where λ_o is the annual growth rate. Moreover, we adopt $\lambda_o = 1.005$, which is slight below about the population growth rate of Europe (the initial member countries of the European Commission) from 1820 throught the present, and $\lambda_o = 1.0003$, which is the Japanese rate since the 1980s. Finally, we study the cases of $\lambda_o = 1.015$ and 1.02 purely for the sake of comparison.

With the data limitation, we can only use a rather rough estimate for the concavity of the differentiated good production, x^α. By (9) and (11), $1/\alpha$ is the markup rate (the ratio between the price and the marginal cost). According to Basu (1995), the aggregate production function is almost linearly homogeneous ($\alpha \approx 1$). As Norrbin (1993) points out, the markup rate can be around 1.05; also see Johnson and Williams (1996). With these considerations, we set $\alpha = 0.95$.

For β, the capital distribution share, we examine cases of $\beta = 0.03$, 0.05, and 0.07. This refects the following considerations. Our macroeconomic production function, (3), can be written in the Cobb-Douglas form of which the capital distribution share is either β or $\alpha\beta$,

$$\frac{rK}{Y} = \beta \text{ or } \alpha\beta; \tag{56}$$

see the appendix for a proof. In the days of the first and second industrial revolutions, this may be identified with the capital adopted in the manufacturing sector; if this is denoted as K_M, $K_M = K$. Since α is assumed to be almost 1, β can be interpreted as the share of the manufacturing sector's capital in GDP, or $\beta \approx rK/Y$. This may be interpreted as the product of the manufacturing sector in GDP, which may be denoted as $\beta_M = Y_M/Y$, and the capital distribution rate in the manufacturing sector, $\theta_M = rK_M/Y_M$; i.e.,

[2]This value is calcurated by using Broadberry, Campbell, Klein, Overton, and Leeuwen (2011, Table 10) for 1760-1830 and Maddison (2023) for 1831-1890.

$\beta = \beta_M \theta_M$. According to Lee and Rhode (2018, Figure ???), in the United States, the share of the manufacturing sector in GDP is between 19 percent in the 1850s and 28 percent in the 1920s; i.e., $\beta_M \in [0.18, 0.30]$. Moreover, Johnson (1954, Table 1) reports that from the 1850s through 1920s, the captal distribution rate (or the sum of entrepreneurial property income, interest, and corporate profits) is about 20 percent, i.e., $\theta_M = 0.2$. These facts suggest $\beta = 0.05$ may be the best point estimate. For the sake of comparison, we present results for $\beta = 0.03$, 0.05, 0.06.

As (5) shows, μ relates income Y_t in period t to capital $K_{t+1} = \mu Y_t$ in the next period, $t+1$, without taking capital depreciation into account. It is usually assumed more realistically that capital K_{t+1} is the sum of saving sY_t and capital left after depreciation $(1-\delta)K_t$, i.e.,

$$K_{t+1} = sY_t + (1-\delta)K_t = \mu Y_t. \tag{57}$$

Since $r_t K_t / Y_t \approx \beta$, this relationship gives rise to

$$\mu \approx s + \frac{(1-\delta)\beta}{r}, \tag{58}$$

where s and δ are, respectively, the saving rate and the depreciation rate. The depreciation rate is generally considered to be around 10 percent; for example Pickety (2014) argues that it is below 10 percent during the 19th century while it is above 10 percent after that. In the Japanese government currently assume about 8 percent (Hayashi and Inoue, 1991, and Bank of Japan, 2025). It is generally perceived that the depreciation rate is generally considered to be around 10 percent; for example, Pickety (2014) argues that it is below 10 percent during the 19th century while it is above 10 percent after that. Our results suggest show that the average wave length is affected little by the depreciation rate. For these reasons, we adopt $\delta_o = 0.08$. For the interest rate, we adopt the long-run rate, which is about $r_o = 4\%$. Assuming the saving rate is $s = 0.2$, we have $\mu_o = 0.2 + (1-\delta_o)/r_o$. In all the cases in which $\beta = 0.3$, 0.5, and 0.7, and in which the lengths of the model's single period are 8, 9, and 10 years, the values of the depreciation rate are about $\mu = 0.25$.

Following Yano and Furukawa (2023), we assume that the length of the model's single period is *eight* to *nine years* and call a time period from one big technology burst to the next an industrial revolution cycle if its length is about 90 years to 130 years. On average, as the table shows, industrial revolution cycles are observed when the population growth rates are not too high and not too low; in the case of $\beta = 0.05$, the average wave length of big technology bursts is 90 years to 130 years if the population growth rate are between 0.75 percent to 1.2 percent. Either below or above this range of population growth rate, the average wave length is shorter than 90 years.

These findings are consistent with the three industrial revolutions we have experienced in the past. As is discussed above, the first industrial revolution and the first half of the second industrial revolution were led by the UK.

The driving force of the first industrial revolution was British inventor's inventions such as the spinning jenny introduced in 1764-1765 by James Hargreaves and the steam engine introduced in 1776 by James Watt. The first half of the second industrial revolution was led by the steal converter invented in 1855 by a British inventor, Henry Bessemer. As is discussed above, during this period from the 1760s through the 1880s, the UK population growth rate was about 0.9 percent.

Since the latter half of the second industrial revolution, starting in the 1890s, world innovation has been led by US inventions. The latter half of the second industrial revolution was led by a series of electric appliances invented by Thomas Edison and automation for car production invented by Henry Ford. Since then, many new innovation has been initiated by US inventors. The third industrial revolution was initiated by the introduction of personal computers and related softwares introduced by US technology companies such as Apple, IBM, and Microsoft. Since then, it has continuously been led by US companies such as those called GAFA (Google Apple, Facebook and Amazon). During this period since the 1890s, the US population growth rate is about 1.2 percent.

Our results shows that an average industrial revolution cycle does not occur in an economy with lower population growth rates. Since the modern market economy was established in the 18th century, except the UK and the US, there has never been a region in which long-term population growth rates are higher than 1 percent. This might explain an industrial revolution has never been lead in a region outside of the US and the UK.

5 Concluding Remarks

Although our model is highly streamlined, it suggests that pupulation growth might be a key to an industrial revolution. As Yano and Furukawa (2023) and Yano (2025) explain, industrial revolution cycles of this study may be attributable to the interaction between the scarcity of technologies (or the need for innovation) and what Yano (2009) calls market quality.

If population grows continuously (and exogenously), it increases the scarcity for technologies. As the scarcity rises, the demand for new inventions also rises. Since the ownership of new inventions are protected, this extra demand leads to the establishment of many new technologies firms. As Boldrin and Levine (2005, 2008) point out, however, this has both good and bad sides. The bad side is: Because the monopolistic use of the new technologies are guaranteed for those monopolistic technology firms, it lowers market quality. Once those firms enters the market, however, their monopolistic power will be reduced. As the scarcity of new technologies is resolved, the development of new technologies slows, although new entry will continue, thereby raising market quality. During this period, population will continue to grow, which will raise the scarcity of technologies again.

The present study shows that this balance of the scarcity of technologies and market quality gives rise to industrial revolution cycles. Our result suggests that one percent average population growth may be a key to this cycle and an industrial revolution.

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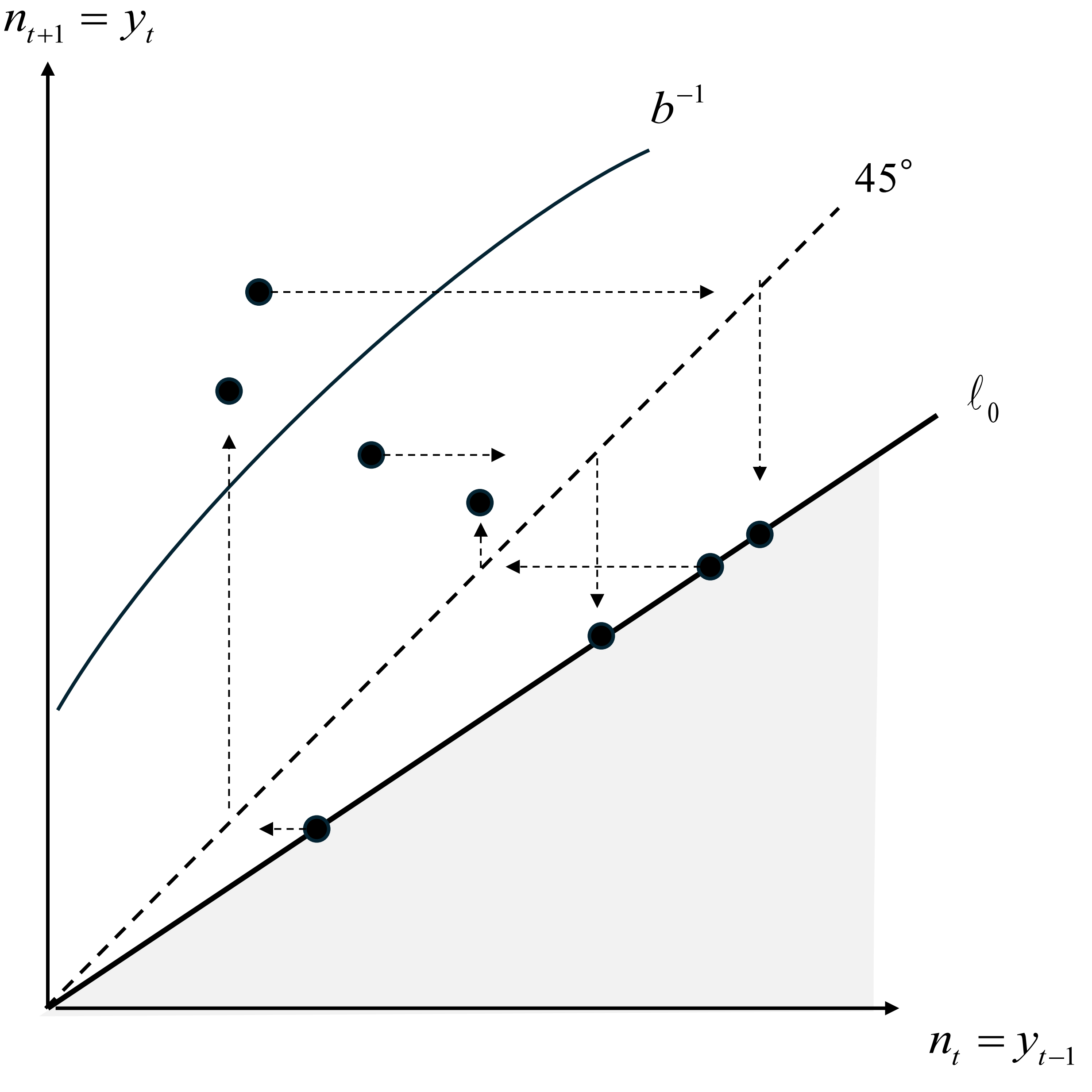

Figure 1: Two-dimensional Constrained Dynamics

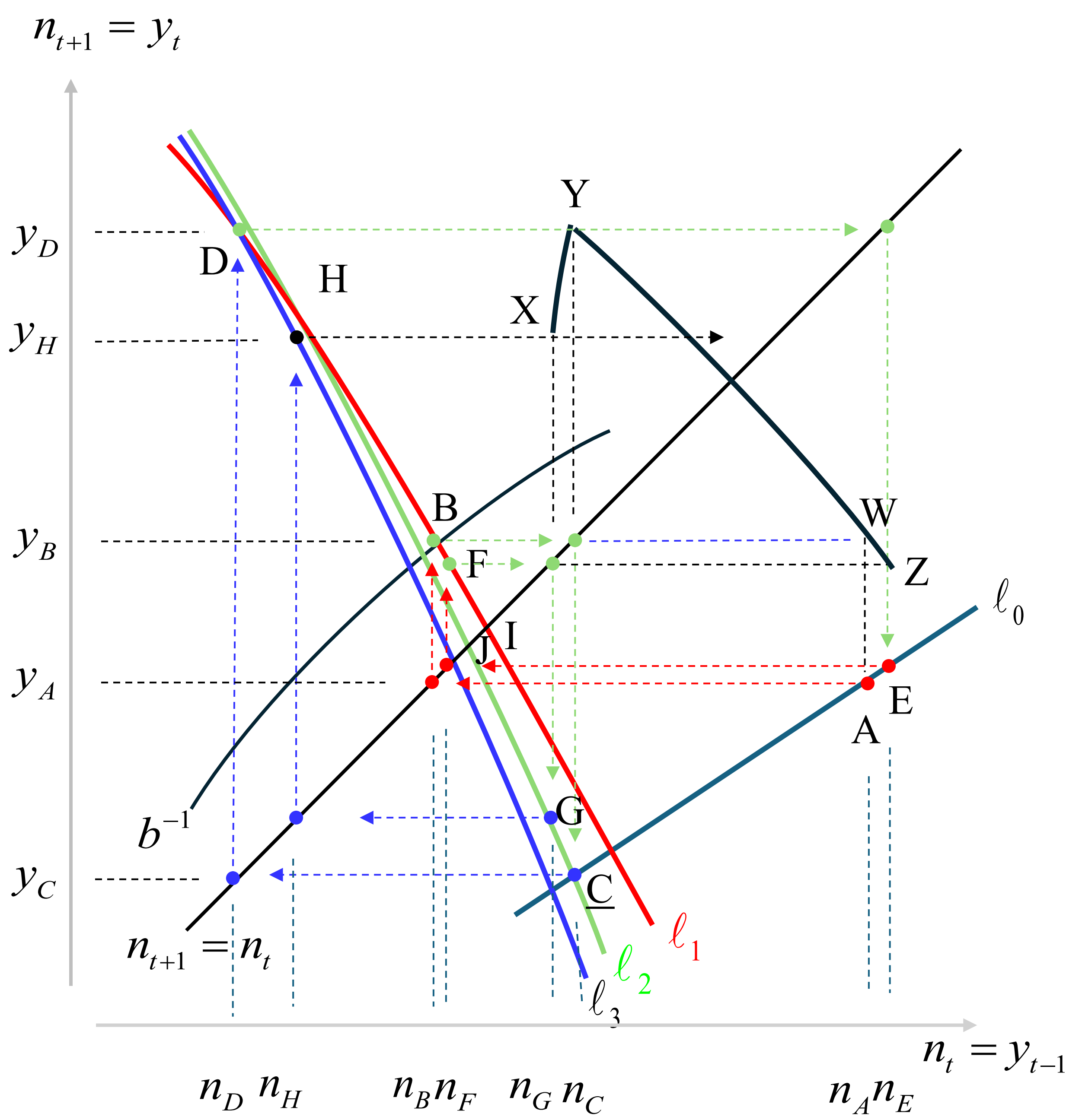

Figure 2. Two-dimensional Ergodic Chaos

population growth rate per year	length of the model's single period	population growth rate per period	capital share β		
		λ	0.03	0.05	0.07
0.03%	8 years	1.0024	67.294751	67.900187	71.87135
	9 years	1.0027	75.700227	76.582709	79.716563
	(10 years)	1.003	84.631009	85.26603	88.542589
0.50%	8 years	1.04	66.088393	72.02665	78.554595
	9 years	1.045	76.995466	85.146641	96.071734
	(10 years)	1.05	85.984523	94.41979	110.87704
0.75%	8 years	1.06	73.971336	83.507307	94.462156
	9 years	1.0675	83.441498	97.18173	113.15062
	(10 years)	1.075	102.06165	117.67475	125.45477
1%	8 years	1.08	86.711468	93.622001	98.667982
	9 years	1.09	102.59918	111.06997	91.911765
	(10 years)	1.1	119.10434	111.17287	90.991811
1.20%	8 years	1.096	91.785223	97.276265	76.284924
	9 years	1.108	106.58456	87.44656	77.512703
	(10 years)	1.12	101.83299	88.378259	81.599347
1.50%	8 years	1.12	81.466395	70.702607	65.279478
	9 years	1.135	78.98201	72.998621	65.852052
	(10 years)	1.15	80.366471	72.684983	75.024383
2%	8 years	1.16	60.051043	57.28197	56.713455
	9 years	1.18	65.335753	62.128952	58.59375
	(10 years)	1.2	65.772165	62.5	N/A

Table 1. Average Wave Length of Industrial Revolution Cycles

$\mu=0.25$